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# *On some Invariant Scrolls in Collineations which leave a Group of Five Points Invariant.*

By Virgil Snyder.

The quadric surfaces which are left invariant by cyclical collineations have been quite exhaustively studied.*

Another quite simple series of scrolls, namely, certain ones contained in a linear congruence, have equally interesting properties, but have not been heretofore considered, except one form which was noticed by Ameseder. The following note will be restricted to such surfaces. There are six collineations which are of essentially different type that project a set of five points into themselves without leaving every point invariant.

Using the notation of substitution-groups, these six collineations may be represented as follows:

$$
\begin{aligned}
T_2 &\equiv (A_1\,A_2)(A_3)(A_4)(A_5),\\
T_3 &\equiv (A_1\,A_2)(A_3\,A_4)(A_5),\\
T_4 &\equiv (A_1\,A_2\,A_3)(A_4)(A_5),\\
T_5 &\equiv (A_1\,A_2\,A_3)(A_4\,A_5),\\
T_6 &\equiv (A_1\,A_2\,A_3\,A_4)(A_5),\\
T_7 &\equiv (A_1\,A_2\,A_3\,A_4\,A_5).
\end{aligned}
$$

It is not necessary to count the number of ways in which any $T_k$ may be represented; $T_k$, $T_m$ are independent of each other in the sense that each leaves a characteristic configuration invariant, but all the invariant forms which arise from permutation of the elements in a particular $T_k$ are projectively equivalent.

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*J. Lüroth, "Das Imaginäre in der Geometrie und das Rechnen mit Würfen," Math. Annalen, vol. XI, p. 84, and "Ueber cyclisch-projectivische Punktgruppen in der Ebene und im Raume," ibid., vol. XIII, p. 304. A. Ameseder, "Theorie der cyclischen Projectivitäten," Wiener Sitzungsberichte (Math. natw. Classe), vol. XCVIII, IIa, p. 290, and "Die Quintupellage collinearer Räume," ibid., p. 588. R. Sturm, "Ueber Collineationen und Correlationen welche Flächen 2 Grades oder cubische Raumcurven in sich selbst transformieren," Math. Ann., vol. XXV, p. 465. H. Schroeter, "Cyclisch projective Punktquadrupel," Math. Ann., vol. XX, p. 231. H. Küppers, "Collineation durch welche fünf Punkte des Raumes . . . ." Diss. Münster, 1890.

These collineations will be considered in turn.

Let $T_2 \equiv (A_1 A_2)(A_3)(A_4)(A_5)$ be the symbol of a collineation which leaves the points $A_3$, $A_4$, $A_5$ invariant, and interchanges the points $A_1$, $A_2$ involutorially. The plane $A_3 A_4 A_5 \equiv \omega$ remains invariant, and likewise the line $(A_1 A_2) \equiv u$. The intersection of $\omega$ and $u$ remains fixed; hence, every point of $\omega$ is invariant. The other invariant point on $u$ is the harmonic conjugate of $(u, \omega)$ with regard to $A_1 A_2$.

The scrolls having $u$ as an $m$-fold directrix, and any line $v$ in $\omega$ which does not cut $u$ as a double directrix, the two lines which issue from any point of $v$ passing through a pair of conjugates of $u$, will go into themselves; the generators passing through a pair of conjugate points simply interchange; likewise, tangents belonging to positive and negative complexes simply interchange, and, consequently, the asymptotic lines belonging to positive and negative complexes also; hence, *the harmonic homology $T_2$ leaves a series of scrolls of the symbol* $(2, m)$ *invariant in such a way that positive and negative asymptotic lines are interchanged.*

$T_3 \equiv (A_1 A_2)(A_3 A_4)(A_5)$. This form leaves every point of each of two skew lines $u$, $v$ invariant, and is known as the axial involution (Salmon-Fidler, vol. I, p. 132). The point corresponding to any point $A$ lies on the line joining $A$ to $u$ and $v$ and is the harmonic conjugate of $A$ with regard to these two points of intersection, hence, *every scroll having $u$, $v$ for directrices is left invariant by $T_3$ in such a way that every asymptotic line goes into itself, the two points in which it intersects each generator being interchanged.*

$T_4 \equiv (A_1 A_2 A_3)(A_4)(A_5)$. Here space is in triad position. The plane $\omega$, containing $A_1 A_2 A_3$, is invariant and in it the point in which the line $A_4 A_5 \equiv u$ pierces it, hence, $u$ has only invariant points. There is an invariant line in $\omega$ skew to $u$; call it $v$. The points on $v$ are in triads; hence, the invariant points must be imaginary (Lüroth, l. c. first paper); call them $R$ and $S$.

Let $X_1$, $Y_1$ be any two points in space; they generate two triads $X_1 X_2 X_3$, $Y_1 Y_2 Y_3$. Through these six points a twisted cubic curve $c_3$ can be drawn, which goes into itself. The points on $c_3$ lie in triads; hence, two imaginary invariant points lie upon it. All the self-corresponding points lie on $u$ and $v$. $c_3$ cannot cut $v$, for the plane of each triad passes through $v$, and in that case each plane through $v$ would cut $c_3$ in more than three points.

Hence, $c_3$ cuts $u$ in two imaginary points $OP$. The points $OPRS$ are the vertices of a principal tetrahedron of $T_4$. The Reye complex of lines joining corresponding points breaks up into two special linear complexes with $u$, $v$ for axes. The line joining any two non-coincident corresponding points must cut $v$.

The osculating planes of $c_3$ at $O$ and $P$ must also remain invariant; hence, they are the planes $(Ov)(Pv)$. The tangents at these points are also invariant; choose $O$, $P$ such that the tangents are $OR$, $PS$. Now, consider the scroll $\Sigma$ generated by all the lines which cut $u$, $v$ and $c_3$. This will be a quartic scroll having $u$ for a triple directrix and $v$ for a simple one. The three lines which issue from the same point of $u$ cut $v$ in the points of a triad. The pinch-points are $O$, $P$, each counted twice. Let $Z$ be any point on the surface $\Sigma$; it will generate a triad $Z_1$, $Z_2$, $Z_3$. Determine on a second generator $g$ not belonging to the first triad, a point $W$ such that the four points $gu$, $gv$, $gc_3$, $W$ have the same anharmonic ratio as the points $g'u$, $g'v$, $g'c_3$, $Z$; $Z$ lying on $g'$. Then $W$ will generate a triad, and the locus of $W$, as $g$ takes all positions on $\Sigma$, will be another twisted cubic lying on the same surface. Dually, let $\alpha$, $\beta$ be any two planes; they will generate two triads $\alpha_1\alpha_2\alpha_3$, $\beta_1\beta_2\beta_3$. The first triad will pass through a fixed point $U_1$ on $u$, and the second through another, $U_2$. These planes cut $v$ in two triads; they uniquely determine a cubic torse $\varkappa_3$, all of whose planes are arranged in triads, and will, therefore, have two imaginary invariant planes $\phi_1$, $\phi_2$ through $v$. The points of contact are also invariant and lie on $u$, hence, they must be the points $O$, $P$, and the invariant planes are $(Ov)$, $(Pv)$. These conditions fixed $c_3$, hence, $c_3$ is self-dual, and all the generators of $\Sigma$ lie in the osculating planes of $c_3$; these planes are also tangent planes to $\Sigma$, hence, $c_3$ is an asymptotic line on the surface. The other lines, loci of the points $W$, are the remaining asymptotic lines; hence, $T_4$ *transforms* $\infty^6$ *quartic scrolls of the type* $[3, 1]$ *into themselves in such a way that the cubic asymptotic lines go into themselves.*

The Weddle surface $\psi$, locus of the vertex of a quadric cone passing through two triads $X$, $Y$, goes into itself in such a way that the cubic lying in any plane through $v$ goes into itself. The curve $c_3$ is an asymptotic line on this surface, hence, $\Sigma$ and $\psi$ touch each other along $c_3$. The $\infty^6$ surfaces $\psi$ all pass through the line $v$, and on them it is a single line.

Each plane cubic cuts $v$ in the points of a triad, and the points in its plane on $c_3$ define another triad.

The 12 Kummer surfaces which are determined by the two triads $(A_1 A_2 A_3)$, $(B_1, B_2, B_3)$ of nodes go into each other in sets of fours; it is always possible to choose $B_1$ such that the triads form a quadruple involution on $c_3$; they lie in triple involution in any case. All 12 surfaces of the group are in these cases tetrahedroids, projectively equivalent to Fresnel's wave surface. The surface $\psi$ also acquires two new lines.*

$T_5 \equiv (A_1 A_2 A_3)(A_4 A_5)$. Since $T_5^2 = T_4$ and $T_5^3 = T_2$, it follows that only those configurations which are left invariant by both $T_4$ and $T_2$ are invariant in $\mathrm{T}_5$. No simple scrolls except hyperboloids remain invariant, and in these the two systems of generators are interchanged. $T_5^6 = 1$, so that space lies in sextuple position, and it might be supposed that a twisted cubic could be passed through a sextuple, but as the points of every sextuple lie in pairs on three lines through an invariant point on account of $T_5^3 = T_2$, hence, no other cubic through these points is possible.

$T_6 \equiv (A_2 A_2 A_3 A_4)(A_5)$. This case has already been considered in detail in the article by Schroeter mentioned above. The scrolls defined by secants of a pair of rays in involution and the invariant quartic curve, break up into two quadrics, and there are no others. The secants of this quartic and the invariant lines form the second generation of the same quadrics.

$T_7 \equiv (A_1 A_2 A_3 A_4 A_5)$. Let it be supposed that the five points are so arranged that no four lie in a plane, then two skew lines $u$, $v$ remain invariant, and on each are ranges in quintuple. Then $u$, $v$ have two sets of imaginary invariant points, $O$, $P$ on $u$; $R$, $S$ on $v$. A twisted cubic $c_3$ is uniquely determined by passing through the five points of a quintuple and having $v$ for a chord; $c_3$ will go into itself by $T_7$. On it there are two invariant points which must be $R$, $S$. Consider a scroll $\Sigma$ generated by lines cutting $u$, $v$, $c_3$; $\Sigma$ will be of the fourth order, having $v$ for a triple directrix and $u$ for a simple one; it remains invariant in $T_7$. Now, consider, dually, a plane $\alpha$ and construct the torse $\varkappa_3$ determined by the five planes $\alpha_r$ and having $u$ for an axis. This torse remains invariant, and has two imaginary invariant planes $(uR)$, $(uS)$. The points of contact also remain invariant; hence, $c_3$ is self-dual and identical with $\varkappa_3$. Finally, regard $c_3$ as directrix of a linear complex; the surface $\Sigma$ being self-dual and $c_3$ lying upon it, it appears that the generators of $\Sigma$ all lie in the corresponding osculating planes of $c_3$; hence $c_3$ is an asymptotic line of the surface. By choosing another point on any generator, and passing a corresponding $c_3$ through it, all the asymptotic lines can be obtained. They all pass through the same points $R$, $S$, which are, therefore, pinch-points; they have the same osculating planes $(Su)$, $(Ru)$, which are the torsal planes, and, finally, all the same tangents, the limiting generators of the surface. Another set of scrolls similar to these can be obtained by interchanging $u$ and $v$. *$T_7$ leaves $\infty^3$ quartic scrolls of the type* $[3, 1]$ *invariant in such a way that the cubic asymptotic lines remain invariant.*

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* See J. I. Hutchinson, "Note on the Tetrahedroid," in the Bulletin of the American Mathematical Society, vol. IV, p. 327, and "On a Special Form of a Quartic Surface," Annals of Mathematics, vol. XI, p. 158.

The general type of the equations of these scrolls, the invariant lines being chosen for $(x, y)(z, w)$ respectively are

$$\text{In } T_2: \quad x^2\phi_m(z, w) + y^2 f_m(z, w) = 0,$$

$\phi$, $f$ being binary quantics.

$$\text{In } T_3: \quad F(\lambda, \mu) = 0, \quad \lambda = x:y, \quad \mu = z:w.$$

In $T_4$ and $T_7$ the invariant scroll has the same form; usually the asymptotic lines on a $[3, 1]$ scroll are of order 6,[*] here they are of order 3, and the pinch-points coincide in pairs, and are imaginary. Conversely, every $[3, 1]$ having two double imaginary pinch-points has asymptotic lines of the third order.

Any $[3, 1]$ scroll can be expressed in the form

$$zx(x^2 + 3cy^2) + wy(y^2 + 3cx^2) = 0.$$

The values of $\lambda$, which correspond to the pinch-points, are determined by the equation

$$3c^2 - 1 = c\left(\lambda^2 + \frac{1}{\lambda^2}\right),$$

which is to have two double imaginary roots. The admissible values of $c$ are $-1$ and $-\frac{1}{3}$, which go into each other by interchanging $x$ and $y$,

$$\Sigma \equiv zx(x^2 - 3y^2) + wy(y^2 - 3x^2) = 0.$$

The asymptotic lines are determined by this equation and

$$x\frac{\partial\Sigma}{\partial z} + \varkappa w\frac{\partial\Sigma}{\partial x} = 0.$$

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[*] V. Snyder, "Asymptotic Lines on Ruled Surfaces," Bulletin Amer. Math. Society, vol. V, p. 343.

The projection of these curves on the plane $z = 0$ gives

$$\left[(x^2 - 3y^2)\,x + \sqrt{\frac{\varkappa}{3}}\,(x^2 + y^2)\,w\right]\left[(x^2 - 3y^2)\,x - \sqrt{\frac{\varkappa}{3}}\,(x^2 + y^2)\,w\right] = 0\,.$$

These curves all touch each other at the conjugate point $x = 0$, $y = 0$ projections of $R$ and $S$. By changing the sign of the parameter $\varkappa$, all the lines become imaginary; hence, all the asymptotic lines of $\Sigma$ belong to right-hand complexes. These two cubics make up a sextic, and their points of intersection with any generator are harmonic as to $u$ and $v$; hence, they go into each other by $T_3$. The generators from every point of $x$, $y$ are all real.

If five points $A_1 \ldots\ldots A_5$ be chosen arbitrarily in space, no point $A_6$ can be found such that $T^6 = 1$, where $T \equiv (A_1 A_2 A_3 A_4 A_5 A_6)$; for $T^2 \equiv (A_1 A_3 A_5)(A_2 A_4 A_6)$ and $T^3 \equiv (A_1 A_4)(A_2 A_5)(A_3 A_6)$. $T^2$ is identical with $T_4$ and $T^3$ either with $T_3$ or $T_2$. It cannot be the latter, as the lines joining $A_1 A_4$, $A_2 A_5$ would then meet in a point $O$, which is contrary to the hypothesis that the points are chosen arbitrarily. Hence,

$$T^3 \equiv T_3, \quad T^2 \equiv T_4,$$

and only those configurations can remain invariant which are left so by both these transformations. In $T_3$ every point of $u$ and $v$ remains invariant, and every line joining corresponding points, as $A_1 A_4$, cuts both. In $T_4$ the plane of each triple passes through $v$, and as $A_1 A_4$ also cuts $v$, and similarly $A_2 A_5$, $A_3 A_6$ all of the points lie in a plane, contrary to hypothesis.

These planes all lie in the same pencil, and cut conics (containing the six points) from a fixed quadric surface, which, when the surface is projected into a hyperboloid of revolution, become parallel. $T$ is now a rotation of 60° about the axis of this hyperboloid.

CORNELL UNIVERSITY, *April* 7, 1900.